Mellin amplitudes for $AdS_5 \times S^5$

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Integrable, but not solved yet!

Planar $\mathcal{N} = 4$ SYM should be under full analytic control. But in practice, computations are hard and we lack efficient algorithms to compute most observables.

Dramatic illustration:

Correlators $\langle \mathcal{O}_{p_1}(x_1) \ldots \mathcal{O}_{p_n}(x_n) \rangle$ of one-half BPS local operators,

$$\mathcal{O}^{I_1 \ldots I_p}_p(x) = \text{Tr} \ X^{\{I_1 \ldots X^{I_p}\}}(x), \quad I_k = 1, \ldots 6.$$  

Trivial for $n = 2, 3$ but wild for $n \geq 4$.

We should not be content till we have found an efficient way to compute these correlators, for all values of the 't Hooft coupling $\lambda$. 

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Mellin amplitudes
Embarrassingly, they are very hard even for $\lambda \to \infty$ (classical sugra limit)!

Classical sugra is a complicated non-linear theory.

Only very few holographic correlators are known, despite heroic efforts:

1. Three cases with $p_i = p$, namely $p = 2, 3, 4$
   Arutyunov Frolov, Arutyunov Dolan Osborn, Arutyunov Sokatchev

2. The class $p_1 = n + k$, $p_2 = n - k$, $p_3 = p_4 = k + 2$
   (“next-to-next-to-extremal” cases)
   Uruchurtu

Is there a better way?
In this talk, I will describe an explicit formula for the 4pt half-BPS correlator in the sugra limit, for arbitrary \( p_i \).

- **New tool:** **Mellin representation** of CFT correlators
  Mack, Penedones, …

- **New idea:** **on-shell “bootstrap” approach**, inspired by modern methods for perturbative scattering amplitudes. The diagrammatic expansion hides the true simplicity of the final on-shell result. Instead, we will directly fix the final answer by imposing consistency conditions.
Kinematics

Eliminate $SO(6)$ indices by contracting them with a null vector

$$O_p(x, t) = t_{I_1} \ldots t_{I_p} O^{I_1 \ldots I_p}_p(x), \quad t \cdot t = 0.$$

For simplicity, I will restrict to $p_i = p$ in most of the talk. Generalization to unequal $p_i$ at the end.

Using bosonic subgroup $SU(2, 2) \times SU(4) \subset PSU(2, 2|4)$,

$$\langle O_p(x_1, t_1) \ldots O_p(x_4, t_4) \rangle = \left( \frac{t_{12} t_{34}}{x_{12}^2 x_{34}^2} \right)^p G(U, V; \sigma, \tau),$$

where $x_{ij} = x_i - x_j$, $t_{ij} = t_i \cdot t_j$ and

$$U = \frac{(x_{12})^2 (x_{34})^2}{(x_{13})^2 (x_{24})^2}, \quad V = \frac{(x_{14})^2 (x_{23})^2}{(x_{13})^2 (x_{24})^2},$$

$$\sigma = \frac{t_{13} t_{24}}{t_{12} t_{34}}, \quad \tau = \frac{t_{14} t_{23}}{t_{12} t_{34}}.$$

Note that $G(U, V; \sigma, \tau)$ is a polynomial of degree $p$ in $\sigma$ and $\tau$,

$$G(U, V; \sigma, \tau) = \sum_{0 \leq m + n \leq p} \sigma^m \tau^n G^{(m,n)}(U, V).$$
Review of traditional method

To leading $O(1/N^2)$ order,

$$\mathcal{A}_{\text{sugra}} = \mathcal{A}_{\text{exchange}} + \mathcal{A}_{\text{contact}}$$

- **External legs**: bulk-to-boundary propagators $K_{\Delta_i}(x_i, Z)$.
- **Internal legs**: bulk-to-bulk propagators $G_{\Delta,\ell}(Z, W)$.
- **Vertices**: effective action obtained by KK reduction of IIB sugra on $S^5$. 

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Mellin amplitudes
Contact diagrams with no derivatives are knowns as $D$-functions,

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} = \int [dZ] \ K_{\Delta_1}(x_1, Z) K_{\Delta_2}(x_2, Z) K_{\Delta_3}(x_3, Z) K_{\Delta_4}(x_4, Z).$$

$D_{1111}$ is the scalar box integral in $4d$; the higher $D$-functions are obtained by taking derivatives in $x^2_{ij}$.

These are the basic building blocks. Somewhat miraculously, all exchange diagrams that occur in $AdS_5 \times S^5$ can be written as finite sums of $D$-functions. D’Hoker Freedman LR

Example: scalar exchange.
For external dimensions $\Delta_i = p$, internal dimension $\Delta$ (always even),

$$A_{p p p p}^{\Delta, \ell=0}(x_1, x_2, x_3, x_4) = \sum_{k=\Delta/2}^{p-1} a_k |x_{12}|^{-2p+2k} D_{k k p p}(x_1, x_2, x_3, x_4)$$
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Sugra modes exchanged in the holographic one-half BPS four-point function. Twist cut-off: $\tau \leq \min\{p_1 + p_2, p_3 + p_4\} - 2$ for an $s$-channel exchange.

Heroic calculation of quartic couplings by Arutyunov and Frolov: 15 pages just to write them down.

Proliferation of diagrams and tedious combinatorics make this very hard already at $p_i \sim 4$.

With rearrangements, some hopeful hints of simplication: 1, 4, 14 $D$s for $p = 2, 3, 4$, but answers completely non-transparent.
CFT correlators as AdS scattering amplitudes

CFT correlators are the best analog we have in AdS for an S-matrix. They are on-shell objects. Mellin space makes this analogy manifest.

Scattering in AdS

Penedones, Fitzpatrick Kaplan Penedones Raju, Paulos
Mellin representation of CFT correlators

\[ G(x) = \langle O_1(x_1)O_2(x_2) \ldots O_n(x_n) \rangle_{\text{conn}} = \int [d\delta_{ij}] M(\delta_{ij})(x_{ij}^2)^{-\delta_{ij}}. \]

The integration variables obey the constraints

\[ \delta_{ij} = \delta_{ji}, \quad \delta_{ii} = -\Delta_i, \quad \sum_{j=1}^{n} \delta_{ij} = 0 \]

\( M(\delta_{ij}) \) is the so-called reduced Mellin amplitude [Mack].

The operator product expansion

\[ O_1(x_1)O_2(x_2) = \sum_i \frac{C_{12i}}{(x_{12}^2)^{\Delta_1+\Delta_2-\Delta_i}} (O_i(x_2) + \text{descendants}) \]

implies that \( M(\delta_{ij}) \) has simple poles at

\[ \delta_{12} = \frac{\Delta_1 + \Delta_2 - \Delta_i + 2n}{2} \]

with residue \( \sim C_{12i} \langle O_i O_3 \ldots O_n \rangle \). Poles with \( n > 0 \) come from descendants.

\( M \) is a meromorphic function! It has factorization properties somewhat analogous to tree-level scattering amplitudes in flat space.
The constraints are solved by

\[ \delta_{ij} = p_i \cdot p_j , \]

with \( \sum_i p_i = 0 \) and \( p_i^2 = -\Delta_i \).

In 4pt case, two independent variables. We can use the Mandelstam invariants

\[ s = -(p_1 + p_2)^2 , \quad t = -(p_2 + p_3)^2 , \quad u = -(p_1 + p_4)^2 , \quad s + t + u = \sum_i \Delta_i . \]

- \( M(s,t) \) has the usual crossing symmetry properties of a 4pt S-matrix.
- Primary operator \( O \) in the \( s \)-channel OPE \( \rightarrow \) infinite tower of poles at
  \[ s_0 = \tau_O + 2n \]
  with polynomial residue in \( t \) (Mack polynomial). \( \tau = \Delta - J \) is the twist.

- Analogous statements in \( t \)- and \( u \)-channel.
Mellin amplitude at large $N$

Now define the Mellin amplitude $\mathcal{M}$ by [Mack]

$$M(\delta_{ij}) = \mathcal{M}(\delta_{ij}) \prod_{i<j} \Gamma[\delta_{ij}].$$

This is the natural definition at large $N$, because the explicit poles in the product of Gamma's account precisely for the double-trace contributions. [Penedones]

For example, $\Gamma(\delta_{12})$ gives poles corresponding to the double-traces

$$\mathcal{O}_{\Delta_1} \partial^J \Box^n \mathcal{O}_{\Delta_2},$$

of twist $\tau = \Delta_1 + \Delta_2 + 2n + O(1/N^2)$. 

In the planar limit, $\mathcal{M}$ contains only single-trace poles.
Mellin amplitudes for Witten diagrams

\[ D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \] has \( \mathcal{M} = 1 \).

The simplification of sugra calculations in Mellin space descends from this fact.

**Contact diagrams** with \( 2k \) spacetime derivatives have \( \mathcal{M} = P_k(s, t) \), polynomial of degree \( k \).

**Exchange diagrams** (in the \( s \)-channel) take the general form

\[
\mathcal{M}_{\Delta, J}(s, t) = \sum_{m=0}^{\infty} \frac{Q_{J,m}(t)}{s - (\Delta - J) - 2m} + P_{J-1}(s, t)
\]

for arbitrary internal dimension \( \Delta \) and spin \( J \).

When specialized to the quantum numbers of \( AdS_5 \times S^5 \), sum over \( m \) truncates at some \( m_{\text{max}} \). This is the translation of the fact that exchange diagrams are **finite** sums of \( Ds \).
This is actually necessary for consistency of the OPE interpretation. Single-trace poles must truncate before they overlap with the double-trace poles from $\Gamma(-s/2 + p)^2$.

A double pole in $s$ corresponds to a $\log U$ term in $G_{\text{conn}}$, and is needed to account for the $O(1/N^2)$ anomalous dimensions of the double-trace operators. But a triple pole cannot arise at this order.
We are learning that in the sugra limit $\mathcal{M}$ is a very constrained function. Our goal is to characterize it from a set of abstract conditions.

Taking stock, our discussion so far implies:

1. **Crossing symmetry**

   \[
   \sigma^p \mathcal{M}(u, t; 1/\sigma, \tau/\sigma) = \mathcal{M}(s, t; \sigma, \tau)
   \]

   \[
   \tau^p \mathcal{M}(t, s; \sigma/\tau, 1/\tau) = \mathcal{M}(s, t; \sigma, \tau)
   \]

2. **Analytic properties:**

   $\mathcal{M}$ has a finite number of simple poles in $s, t, u$ at $2, 4, \ldots 2p - 2$. The residue at each pole is a polynomial in the other variable.
Next, consider the asymptotic behavior for large $s, t$.

Exchange diagrams grow at most linearly, since $J \leq 2$, and

$$M_{\Delta,J}(s,t) = \sum_{m=0}^{m_{\text{max}}} \frac{Q_{J,m}(t)}{s - (\Delta - J) - 2m} + P_{J-1}(s,t).$$

We shall assume that contact diagrams have the same asymptotics. This is true if contact diagrams with at most two derivatives contribute, as it happens in all concrete examples. Arutyunov

An indirect argument is consistency of the flat space limit, which must be dominated by graviton exchange.

In summary,

3. Asymptotics:

$$M(\beta s, \beta t; \sigma, \tau) \sim O(\beta) \quad \text{for } \beta \to \infty.$$
We should finally impose the full constraints of $PSU(2, 2|4)$. In position space, the superconformal Ward identity reads

$$\partial_{\bar{z}}[G(z\bar{z}, (1 - z)(1 - \bar{z}); \alpha\bar{\alpha}, (1 - \alpha)(1 - \bar{\alpha}))|_{\bar{\alpha} \to 1/\bar{z}}] = 0,$$

where we have performed the useful change of variables $U = z\bar{z}$, $V = (1 - z)(1 - \bar{z})$, $\sigma = \alpha\bar{\alpha}$, $\tau = (1 - \alpha)(1 - \bar{\alpha})$.

The solution is

$$G(U, V; \sigma, \tau) = G_{\text{free}}(U, V; \sigma, \tau) + R \ H(U, V; \sigma, \tau),$$

where

$$R = \tau 1 + (1 - \sigma - \tau) V + (-\tau - \sigma\tau + \tau^2) U + (\sigma^2 - \sigma - \sigma\tau) UV + \sigma V^2 + \sigma\tau U^2$$

$$= (1 - z\alpha)(1 - \bar{z}\alpha)(1 - z\bar{\alpha})(1 - \bar{z}\bar{\alpha}).$$

All dynamical information is contained in $H(U, V; \sigma, \tau)$. 
To write the Ward identity in Mellin space, we first define a Mellin amplitude \( \tilde{\mathcal{M}} \) associated to the dynamical \( \mathcal{H} \) function,

\[
\mathcal{H} = \int dsdt\ U^{s/2}V^{t/2-\Delta} \tilde{\mathcal{M}}(s, t; \sigma, \tau) \Gamma\left[-\frac{s}{2} + \Delta\right]^2 \Gamma\left[-\frac{t}{2} + \Delta\right]^2 \Gamma\left[-\frac{\tilde{u}}{2} + \Delta\right]^2,
\]

with \( \tilde{u} = u - 4 \). For comparison,

\[
\mathcal{G}_{\text{conn}} = \int dsdt\ U^{s/2}V^{t/2-\Delta} \mathcal{M}(s, t; \sigma, \tau) \Gamma\left[-\frac{s}{2} + \Delta\right]^2 \Gamma\left[-\frac{t}{2} + \Delta\right]^2 \Gamma\left[-\frac{u}{2} + \Delta\right]^2.
\]

This shift in \( u \) compensates for the lack of crossing of \( R \), giving simple crossing rules for \( \tilde{\mathcal{M}} \),

\[
\sigma^{p-2} \tilde{\mathcal{M}}(\tilde{u}, t; 1/\sigma, \tau/\sigma) = \tilde{\mathcal{M}}(s, t; \sigma, \tau)
\]

\[
\tau^{p-2} \tilde{\mathcal{M}}(t, s; \sigma/\tau, 1/\tau) = \tilde{\mathcal{M}}(s, t; \sigma, \tau).
\]
We can now translate into Mellin space the solution of the Ward identity

\[ G(U, V; \sigma, \tau) = G_{\text{free}}(U, V; \sigma, \tau) + R \mathcal{H}(U, V; \sigma, \tau). \]

The free part has “zero” transform (sum of delta functions), so

4. **Superconformal Ward identity:**

\[ \mathcal{M}(s, t; \sigma, \tau) = \hat{R} \circ \tilde{\mathcal{M}}(s, t; \sigma, \tau) \]

\[ \hat{R} = \tau 1 + (1 - \sigma - \tau) \hat{V} + (-\tau - \sigma \tau + \tau^2) \hat{U} \]
\[ + (\sigma^2 - \sigma - \sigma \tau) \hat{U} \hat{V} + \sigma \hat{V}^2 + \sigma \tau \hat{U}^2 \]

\( \hat{U} m \hat{V} n \) are difference operators acting as

\[ \hat{U} m \hat{V} n \circ \tilde{\mathcal{M}}(s, t; \sigma, \tau) = \]

\[ \tilde{\mathcal{M}}(s - 2m, t - 2n; \sigma, \tau) \times \left( \frac{2p - s}{2} \right)_m^2 \left( \frac{2p - u}{2} \right)_{2-m-n}^2 \left( \frac{2p - t}{2} \right)_n^2 \]

\[ (h)_n = \frac{\Gamma[h+n]}{\Gamma[h]} \] the Pochhammer symbol.
Our solution

These conditions define a very constrained bootstrap problem. Experimentation at low $p$ leads to the conjecture

$$\tilde{M}(s, t; \sigma, \tau) = \sum_{i+j+k = p-2 \atop 0 \leq i, j, k \leq p-2} \frac{C_{pppp} (p-2)^2 \sigma^i \tau^j}{(s - s_M + 2k)(t - t_M + 2j)(u - u_M + 2i)},$$

where $s_M = t_M = u_M = 2p - 2$.

This satisfies all conditions and reproduces the known sugra results for $p = 2, 3, 4$.

As simple as it could be.

We believe this is the unique solution but lack a complete proof.
Different external weights

\[ \langle O_{p_1}(x_1, t_1) \ldots O_{p_4}(x_4, t_4) \rangle = \prod_{i<j} \left( \frac{t_{ij}}{x_{ij}^2} \right)^{\gamma_{ij}^0} \left( \frac{t_{12} t_{34}}{x_{12}^2 x_{34}^2} \right)^L \mathcal{G}(U, V; \sigma, \tau), \]

\[ \gamma_{12}^0 = \frac{p_1 + p_2 - p_3 - p_4}{2}, \quad \gamma_{13}^0 = \frac{p_1 + p_3 - p_2 - p_4}{2}, \]
\[ \gamma_{34}^0 = \gamma_{24}^0 = 0, \quad \gamma_{14}^0 = p_4 - L, \]
\[ \gamma_{23}^0 = p_4 - L - \frac{p_1 + p_4 - p_2 - p_3}{2}. \]

Assuming \( p_1 \geq p_2 \geq p_3 \geq p_4 \), we distinguish two cases:

\[ L = p_4 \quad (p_1 \geq p_2 \geq p_3 \geq p_4) \]
\[ L = \frac{p_2 + p_3 + p_4 - p_1}{2} \quad (p_1 + p_4 \leq p_2 + p). \]
The general solution

\[ \tilde{M}(s, t; \sigma, \tau) = \sum_{i + j + k = L - 2} \frac{a_{ijk} \sigma^i \tau^j}{(s - s_M + 2k)(t - t_M + 2j)(\tilde{u} - u_M + 2i)}, \]

where

\[ s_M = \min\{ p_1 + p_2, p_3 + p_4 \} - 2 \]
\[ t_M = \min\{ p_1 + p_4, p_2 + p_3 \} - 2 \]
\[ u_M = \min\{ p_1 + p_3, p_2 + p_4 \} - 2 \]

and

\[ a_{ijk} = \left(1 + \frac{|p_1 - p_2 + p_3 - p_4|}{2}\right)^{-1}_i \left(1 + \frac{|p_1 + p_4 - p_2 - p_3|}{2}\right)^{-1}_j \times \left(1 + \frac{|p_1 + p_2 - p_3 - p_4|}{2}\right)^{-1}_k \left(L - 2\right)_{ij} C_{p_1p_2p_3p_4} \]

The next-to-next extremal cases are correctly reproduced.
An independent position space method

Write again

\[ \mathcal{A}_{\text{sugra}} = \mathcal{A}_{\text{exchange}} + \mathcal{A}_{\text{contact}} \]

and expand exchange diagrams as sums of contacts, but do not use the precise vertices. Rather, leave them as undetermined coefficients. Write \( \bar{D} \)-functions as

\[ \bar{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} = R_\Phi \Phi(U, V) + R_V \log V + R_U \log U + R_0 \]

where \( \Phi(U, V) = \bar{D}_{1111} \) is the scalar box diagram, and \( R_{\Phi, U, V, 0} \) are rational functions of \( U \) and \( V \).

The superconformal Ward identity then becomes a set of conditions on the rational coefficient functions of the ansatz for \( \mathcal{A}_{\text{sugra}} \)

\[ R^\text{sugra}_\Phi (z, \bar{z}; \alpha, 1/\bar{z}) = R^\text{sugra}_V (z, \bar{z}; \alpha, 1/\bar{z}) = R^\text{sugra}_U (z, \bar{z}; \alpha, 1/\bar{z}) = 0. \]

This gives set of linear equations for the undetermined coefficients. Overall normalization determined by matching with free field theory,

\[ R^\text{sugra}_0 (z, \bar{z}; \alpha, 1/\bar{z}) = \mathcal{G}_{\text{free}}(z, \bar{z}; \alpha, 1/\bar{z}). \]

We have so far obtained results for the equal weights correlators with \( p = 2, 3, 4, 5 \). The result for \( p = 5 \) agrees with our Mellin formula and with a previous conjecture by Dolan Nirschl Osborn.
Some open questions

- Can one rigorously prove our formula? By showing uniqueness, or by recovering it from some streamlined conventional approach, or from our position space method...

- Is there a more “constructive” way to obtain it using on-shell recursion relations? This would lend itself more easily to the higher $n$-point generalization.

- Is there a supercovariant presentation of $\mathcal{M}$? One-half BPS 4-point functions have a unique structure in superspace. We focussed on the superprimaries. (Likely to be important for higher $n$-point correlators.)

- Can we add $\lambda$ dependence? At strong coupling, $\alpha'$ corrections change the large $s$, $t$ asymptotics. At weak coupling, $O(\lambda^3)$ results in position space available for general $p_i$. Is Mellin a useful language at weak coupling?

- $AdS_7 \times S^4$ works similarly. Stay tuned.

- $AdS_4 \times S^7$ significantly harder. Exchange diagrams are infinite sums of $D$-functions. Still, stay tuned.
The remarkable simplicity of $\mathcal{M}$ is a welcome surprise.

Like the Parke-Taylor formula for tree-level MHV gluon scattering, it succinctly encodes the sum of an intimidating number of diagrams.

Holographic correlators are much simpler than previously understood.

We believe that they should be studied following the blueprint of the modern on-shell approach to perturbative gauge theory amplitudes.